J. Fluid Mech. (1968), *vol.* 32, *part* 4, *pp.* 721–735
Printed in Great Britain

Stability of a Hartmann boundary layer under the influence of a parallel magnetic field

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(Received 16 October 1967)

Stability to infinitesimal disturbances—when a parallel magnetic field is imposed—is investigated for the flow in the boundary layer set up by two-dimensional motion between parallel planes of a viscous, incompressible, electrically conducting fluid under the influence of a transverse magnetic field. The flow is assumed to take place at low magnetic Reynolds number. The usual asymptotic methods are employed for the solution, but, apart from the Tollmien-type power series solution, an exact solution of the inviscid equation is obtained in terms of the hypergeometric function and its analytic continuation. Curves of neutral stability for two-dimensional disturbances are calculated and the results for critical Reynolds number modified to take into account three-dimensional disturbances. The parallel magnetic field is found to have a strong stabilizing influence.

1. Introduction

Since the formulation by Michael (1953) and Stuart (1954) of the linear stability problem for a two-dimensional parallel flow of a viscous, incompressible, electrically conducting fluid in a uniform magnetic field a number of papers have appeared on this topic. The case which has been considered most often is the one where the magnetic field is parallel to the flow direction, but, unfortunately, almost all of the work published on the topic suffers from the perpetuation of an error made originally by Michael (1953) in concluding that Squire's theorem for the non-conducting flows (two-dimensional small disturbances of a parallel flow are the least stable) remains valid. It was fairly recently that Hunt (1966) showed the incorrectness of this and of certain other results which depended on this assumption. In particular, he established that for finite values of the magnetic Reynolds number no parallel magnetic field can completely stabilize a flow which is unstable without it. The parallel magnetic field is therefore seen to be a lesser stabilizing agent than had appeared from the earlier analysis of Stuart (1954), Tarsov (1960) and others, which took into account the two-dimensional disturbances only. All the evidence however still points overwhelmingly to the conclusion that the imposition of a parallel magnetic field enhances the stability.

As far as experiments are concerned, the work done on parallel flows with parallel magnetic field is rather small in volume and nothing as dramatically corroborative of the theory as that achieved by Shubauer & Skramstadt (1947) for the non-conducting case has been achieved. It would be highly desirable now if more experiments were to be carried out to test the hydro-magnetic stability theory and particularly so, since the three-dimensional disturbances have been shown to play a different de-stabilizing role in this case. It is well known though, that the predictions of the linear stability theory in the non-conducting case are not in agreement with the experimental data on the stability of plane Poiseuille flow but do agree closely in the case of the flow in the boundary layer over a flat plate. It would seem probable therefore that success might again be achieved with the boundary-layer flows. It is with this in mind that we have undertaken the analysis that follows. The situation envisaged should be relatively simple to realize in the laboratory and may thus be used for providing a testing ground for the theory.

We consider the flow of an incompressible fluid of uniform conductivity σ, density ρ, kinematic viscosity ν and magnetic permeability μ which takes place between parallel planes under the influence of a uniform magnetic field. The magnetic field has a parallel as well as a transverse component with respect to the flow direction. As in Lock's (1955) analysis of the stability of Hartmann flow, we found that the transverse component of the magnetic field, to the usual degree of approximation, enters the stability problem only by virtue of the deformation it causes in the velocity profile, and not by its direct influence on the disturbances. The problem of the stability of the flow under consideration then reduces to examining the stability of the boundary-layer profile under the influence of the parallel component of the magnetic field.

2. The mathematical formulation

We choose the x_1 axis of a system of rectangular cartesian coordinates (x_1, x_2, x_3) to lie in the flow direction, and choose the origin of co-ordinates so as to make the boundary planes have the equations $x_2 = \pm a$. An externally imposed, uniform magnetic field $(\overline{H}_1, \overline{H}_2, 0)$ is supposed to exist.

Under these circumstances, assuming that the total flux of current in the x_3 direction is zero, it may be shown (see Hartman 1937) that the velocity profile $\overline{U}(x_2)$ and the induced magnetic field $\overline{h}_1(x_2)$ in the x_1 direction (the total magnetic field distribution in the flow region is of the form $(\overline{H}_1 + \overline{h}_1(x_2), \overline{H}_2, 0)$) are given by

$$\overline{U}(x_2) = U_0 \frac{\cosh M^* - \cosh(M^* x_2/a)}{\cosh M^* - 1}, \tag{2.1}$$

$$\overline{h}_1(x_2) = \frac{4\pi\mu\sigma a U_0 \overline{H}_2}{M^*(\cosh M^* - 1)} \{\sinh(M^* x_2/a) - (x_2/a)\sinh M^*\}, \tag{2.2}$$

where U_0 is the speed at the centre and $M^* = \mu \overline{H}_2 a \sqrt{(\sigma/\rho\nu)}$ is a dimensionless parameter known as the Hartmann number. The behaviour of $\overline{U}(x_2)$, the so-called Hartmann profile, is well documented, and its significant feature is that increasing values of M^* rapidly induce in it a boundary-layer behaviour, giving uniform flow in the central portion between the planes and large velocity gradients near the planes. In fact, if say, we expand the velocity distribution in the lower half of the channel, we can write it as

$$\overline{U}(x_2) = U_0[1 - \exp\{(M^*/a)(a + x_2)\}] + O(e^{-M^*}),$$

so that, for only moderately large values of M^*, the boundary layer is, to a very good approximation, exponential in character and is of thickness a/M^*. We, in this paper, shall investigate the stability of this flow for the case when M^* is sufficiently large to allow the velocity distribution to be represented as an exponential ($M^* \geqslant 5$ would be quite adequate). The reader will therefore understand our motivation for choosing the transformation from x_2 to y and the scaling relationships when he comes to equation (2.6).

To analyse the stability we follow the standard procedure and disturb the steady-state velocity and magnetic fields by small periodic velocity and magnetic fields, **v**, **h** respectively, of the form

$$\left.\begin{aligned} \mathbf{v} &= \{v_1(x_2), v_2(x_2), v_3(x_2)\} \\ \mathbf{h} &= \{\hat{h}_1(x_2), \hat{h}_2(x_2), \hat{h}_3(x_2)\} \end{aligned}\right\} \exp\{i(\alpha_1 x_1 + \alpha_3 x_3) + \omega t\}. \tag{2.3}$$

If now the usual manipulations are carried out (see, for example, the survey article by Tatsumi 1962) whereby the disturbed field variables are substituted in the governing equations of magneto-hydrodynamics, the squares and higher powers of small quantities neglected, and $v_1, v_3, \hat{h}_1, \hat{h}_3$ eliminated from the resulting equations, then the equations for the disturbance take the form

$$(U-c)(\phi''-\beta^2\phi) - U''\phi + \frac{i}{\alpha R}(\phi^{\text{iv}} - 2\beta^2\phi'' + \beta^4\phi)$$
$$= A^2\left\{(1+h_1)(\theta''-\beta^2\theta) + \frac{H_2}{i\alpha}(\theta'''-\beta^2\theta') - h_1''\theta\right\}, \tag{2.4}$$

$$(U-c)\theta + \frac{i}{\alpha R_m}(\theta''-\beta^2\theta) = (1+h_1)\phi + \frac{H_2}{i\alpha}\phi', \tag{2.5}$$

where dashes denote differentiation with respect to y, which along with the other dimensionless quantities appearing in equations (2.4) and (2.5) has been obtained by writing

$$\left.\begin{aligned} &y = (a+x_2)/L, \quad L = a/M^*, \quad U(y) = \bar{U}(x_2)/U_0 = 1-e^{-y}, \\ &\alpha = \alpha_1 L, \quad \beta = L\sqrt{(\alpha_1^2+\alpha_3^2)}, \quad c = i\omega/\alpha_1 U_0, \\ &\phi = v_2/U_0, \quad \theta = \hat{h}_2/\bar{H}_1, \quad H_2 = \bar{H}_2/\bar{H}_1, \quad h_1 = \bar{h}_1/H_1, \\ &R = U_0 L/\nu, \quad R_m = 4\pi\mu\sigma U_0 L, \quad A = \sqrt{(\mu/4\pi\rho)}\,H_0/U_0. \end{aligned}\right\} \tag{2.6}$$

To be able to proceed further with the analysis of the formidable problem posed by the equations (2.4) and (2.5) it was found necessary to restrict attention to some special range of the prameters. We shall consider flows of low magnetic Reynolds number and at this stage make the approximation $\alpha R_m \ll 1$. This, as Stuart (1954) and others have pointed out, is a useful approximation, in that as far as experiments in the laboratory are concerned, it is usually more than adequately met.

The approximation $\alpha R_m \ll 1$ allows us to neglect terms in h_1 and also the $(U-c)\theta$ term in (2.5). However, for the parallel magnetic field to have any effect upon stability we require $A^2 R_m$ to remain finite so that $\bar{H}_1$ has to be large. For $A^2 R_m \sim 1$ we find $\bar{H}_2^2/\bar{H}_1^2 \sim M^*/R^*$ where $R^* = U_0 a/\nu$. This gives

$$\bar{H}_2/\bar{H}_1 \sim 1/R^{\frac{1}{2}},$$

R being based upon the boundary-layer width. Now, the values of R in our calculations are very large and we find *a posteriori* that the maximum value of $1/R^{\frac{1}{3}}$ is only about $\frac{1}{300}$. We therefore neglect in (2.4) and (2.5) terms in H_2 compared to terms $O(1)$. θ can then be eliminated between (2.4) and (2.5) giving

$$(U-c)(\phi''-\beta^2\phi)-U''\phi+i\alpha Q\phi = (1/i\alpha R)(\phi^{\text{iv}}-2\beta^2\phi''+\beta^4\phi), \tag{2.7}$$

where

$$Q = A^2R_m = (\mu^2\bar{H}_1^2\sigma a)/(\sigma U_0 M^*). \tag{2.8}$$

Equation (2.7) has to be solved subject to the boundary conditions

$$\phi = \phi' = 0, \quad y = 0; \quad \phi, \phi' \to 0, \quad y \to \infty. \tag{2.9}$$

We will now solve (2.7) for the neutral case, i.e. when c is real. Our problem is reducible very simply to an equivalent two-dimensional problem by writing in (2.7)

$$\alpha R = \beta\bar{R}, \quad \alpha Q = \beta\bar{Q}, \tag{2.10}$$

and this two-dimensional form of the equation (with $\beta = \alpha$) will be considered. Before Hunt's (1966) paper, it was thought that one could deduce from (2.10) that the minimum critical Reynolds number is given by the two-dimensional disturbances. This led most authors to consider the reduced problem only. As we said earlier, Hunt has shown this to be incorrect and we shall modify our results to take into account the effect of three-dimensional disturbances.

The exponential profile considered here also occurs in the boundary-layer flow on a flat plate with constant suction. The methods that follow have largely been adapted from the paper by Hughes & Reid (1965) on the stability of the asymptotic suction profile.

3. Solution of the inviscid equation—analytical method

Ignoring terms of $O(\alpha\bar{R})^{-1}$ we obtain the inviscid form of (2.7) which in our particular case is

$$(1-e^{-y}-c)(\phi''-\alpha^2\phi)+e^{-y}\phi+i\alpha\bar{Q}\phi = 0. \tag{3.1}$$

To be able to satisfy the boundary conditions (2.9) the solution of (3.1) that remains bounded as $y \to \infty$ is required. This solution to be denoted by $\Phi(y)$ provides an asymptotic approximation to a solution of (2.7) valid in a region of the complex y plane determined by the inequality

$$-\tfrac{7}{6}\pi < \arg(y-y_c) < \tfrac{1}{6}\pi, \tag{3.2}$$

and excluding the immediate neighbourhood of the critical point y_c. y_c in this case is determined from the equation $1-c = e^{-y_c}$ so that

$$y_c = -\log(1-c). \tag{3.3}$$

If we now write

$$\delta = +\sqrt{(\alpha^2-iS)} \quad \text{where} \quad S = \alpha\bar{Q}/(1-c), \tag{3.4}$$

and make the following transformations

$$\xi = e^{-y}/(1-c), \quad \phi = \exp\{-\delta(y-y_c)\}f(\xi), \tag{3.5}$$

then (3.1) reduces to the hypergeometric equation,

$$\xi(1-\xi)\frac{d^2f}{d\xi^2}+(2\delta+1)(1-\xi)\frac{df}{d\xi}+(1+iS)f = 0. \tag{3.6}$$

By making the transformation $\xi = e^{-y}/(1-c)$ we have mapped the point $y = \infty$ from the y-plane into the origin in the ξ-plane, and so we require the solution of (3.6) regular in the neighbourhood of $\xi = 0$. This solution is of course a constant multiple of the hypergeometric function $F(p, q; r; \xi)$ given by

$$F(p,q;r;\xi) = \sum_{n=0}^{\infty}\frac{\Gamma(p+n)\,\Gamma(q+n)}{\Gamma(r+n)}\frac{\xi^n}{n!}, \tag{3.7}$$

where $$p = \delta+\surd(1+\alpha^2) \quad q = \delta-\surd(1+\alpha^2), \quad r = 1+2\delta. \tag{3.8}$$

We shall require the inviscid solution $\Phi(y)$ to be normalized by the condition $\Phi(y_c) = 1$ and so we take $f(\xi)$ in the form

$$f(\xi) = \frac{F(p,q;r;\xi)}{F(p,q;r;1)}. \tag{3.9}$$

Gauss's formula gives $$F(p,q;\,r;\,1) = \frac{\Gamma(r)}{\Gamma(p)\,\Gamma(q)}. \tag{3.10}$$

Since $p+q-r = -1 < 0$, the series (3.7) is absolutely convergent for $|\xi| < 1$. $f(\xi)$ therefore provides a valid solution in this region of the ξ-plane.

The point $y = 0$ where the other boundary conditions have to be satisfied has been mapped into the point $\xi = \xi_0 = 1/(1-c) > 1$ for $0 < c < 1$, so we require the analytical continuation of the solution (3.9) into the region $|\xi| > 1$. This is done by cutting the ξ-plane from $\xi = 1$ to $\xi = \infty$ along the positive real axis. To satisfy the inequality (3.2), one has to keep to a path lying below the critical point y_c in the y-plane. This corresponds in the ξ-plane, to taking a path lying above the critical point $\xi = \xi_c = 1$, so the cut in the ξ-plane is supposed to lie below the real axis. The required analytical continuation valid in the region $|1-\xi| < 1$ of the cut ξ-plane can then be written in the form (see Erdelyi, Magnus, Oberhettinger & Tricomi 1953)

$$f(\xi) = 1-(1-\xi)\,F(p+1,q+1;2;1-\zeta)\log(1-\xi)-\sum_{n=0}^{\infty}\frac{A_n(1-\xi)^{n+1}}{(n+1)!}, \tag{3.11}$$

where $$A_n(\delta) = \frac{\Gamma(p+1+n)\,\Gamma(q+1+n)}{\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(n+1)} \times\{\psi(p+1+n)-\psi(q+1+n)-\psi(n+1)-\psi(n+2)\}. \tag{3.12}$$

$\psi(z)$ in the above is the digamma function given by

$$\psi(z) = \Gamma'(z)/\Gamma(z).$$

Equation (3.11) gives the value of $f(\xi_0)$ so long as $|1-\xi_0| < 1$, i.e. $\xi_0 < 2$, this requires $c < \frac{1}{2}$. To keep to a path that lies above the critical point in the ξ-plane, $\log(1-\xi)$ must be taken as $\log|1-\xi|-\pi_i$ for $\xi > 1$.

It may be verified *a posteriori* that the values of c on the neutral curves are small. Following Hughes & Reid (1965) we therefore obtained the values of $f'(\xi_0)$

by expanding in terms of the small parameter $\epsilon = c/(1-c)$. These expressions for $f(\xi_0)$ and $f'(\xi_0)$ are

$$f(\xi_0) = 1+(\epsilon - B_0\epsilon^2)(\log\epsilon - \pi i) + \epsilon g_0(\delta) - \epsilon^2 g_1(\delta) + O(\epsilon^3\log\epsilon), \tag{3.13}$$

$$\begin{aligned} f'(\xi_0) = {} & (1+\log\epsilon - \pi i)(1 - B_0\epsilon + B_1\epsilon^2) - (\log\epsilon - \pi i)(B_0\epsilon - 2B_1\epsilon^2) \\ & + g_0(\delta) - 2g_1(\delta)\epsilon + 3g_2(\delta)\epsilon^2 + O(\epsilon^3\log\epsilon), \end{aligned} \tag{3.14}$$

where

$$\left.\begin{aligned} B_0(\delta) &= \delta - \tfrac{1}{2}iS, \\ B_1(\delta) &= \tfrac{1}{2}\delta + \tfrac{2}{3}\alpha^2 - \tfrac{1}{2}i\delta S - \tfrac{11}{12}iS - \tfrac{1}{12}S^2, \\ g_0(\delta) &= \psi(p+1) + \psi(q+1) - \psi(2) - \psi(1), \\ g_1(\delta) &= B_0\{\psi(p+2) + \psi(q+2) - \psi(3) - \psi(2)\}, \\ g_2(\delta) &= B_1\{\psi(p+3) + \psi(q+3) - \psi(4) - \psi(3)\}. \end{aligned}\right\} \tag{3.15}$$

The argument of the digamma function occurring in (3.15) is complex, and this makes the numerical calculations somewhat laborious. The actual numerical work was done by expanding $g_j (j = 0,1,2)$ in power of δ using Riemann zeta-functions. In the region of interest α and $|\delta|$ are of the same order as c, and S is of the order c^2, and so the terms of order c^3 are given by

$$\alpha^{n_1}\delta^{n_2}\epsilon^{n_3}S^{n_4} \quad \text{when} \quad n_1+n_2+n_3+2n_4 = 3.$$

In the expression for $f(\xi_0)$ and $f'(\xi_0)$ that emerged upon expanding, only terms up to this order were retained, giving

$$\begin{aligned} f(\xi_0) = {} & 1+\{\epsilon - \delta\epsilon^2\}(\log\epsilon - \pi i) - \frac{\epsilon}{\delta} - \frac{\alpha^2\epsilon}{2\delta^2} - \delta\epsilon - \frac{\alpha^4\epsilon}{4\delta^3} \\ & + 2\zeta(2)\delta\epsilon + \{\tfrac{1}{2} - 2\zeta(3)\}\alpha^2\epsilon + \{2\zeta(3)-1\}iS\epsilon + \frac{\alpha^4\epsilon}{8\delta^2} - \frac{\alpha^6\epsilon}{8\delta^4} + \epsilon^2\delta, \end{aligned} \tag{3.16}$$

$$\begin{aligned} f'(\xi_0) = {} & (1+\log\epsilon - \pi i)(1 - \delta\epsilon + \tfrac{1}{2}\delta\epsilon^2 + \tfrac{1}{2}iS\epsilon) - (\log\epsilon - \pi i)(\delta\epsilon - \delta\epsilon^2 - \tfrac{1}{2}iS\epsilon) \\ & - \frac{1}{\delta} - \frac{\tfrac{1}{2}\alpha^2}{\delta^2} - \delta - \frac{\tfrac{1}{4}\alpha^4}{\delta^3} + 2\zeta(2)\delta + \{\tfrac{1}{2} - 2\zeta(3)\}\alpha^2 + \{2\zeta(3)-1\}iS \\ & + \frac{\alpha^4}{8\delta^2} + 2\epsilon\delta - \frac{\alpha^6}{8\delta^4} + \alpha^2\delta + \{2\zeta(4)-1\}\delta^3 + \frac{\alpha^6}{8\delta^3} - \frac{\alpha^8}{16\delta^5} \\ & - 2\{2\zeta(2) - \tfrac{5}{4}\}\epsilon\alpha^2 - 2\{\tfrac{7}{4} - 2\zeta(2)\}iS\epsilon - \tfrac{3}{4}\delta\epsilon^2, \end{aligned} \tag{3.17}$$

where $\zeta(z)$ is the Riemann zeta-function.

The curve of neutral stability for $\bar{Q} = 0{\cdot}02$ (figure 4) was calculated using the expressions (3.16) and (3.17) and was found to be incomplete agreement with the one calculated for the same value using the second method which follows.

4. Solution of the inviscid equation—numerical method

The inviscid equation also has the well-known Tollmien-type solutions given by (see Stuart 1954)

$$\left.\begin{aligned} \phi_A(y) &= (y-y_c)P_A(y-y_c), \\ \phi_B(y) &= P_B(y-y_c) + \frac{(U_c'' - i\alpha\bar{Q})}{U_c'}P_A(y-y_c)\log(y-y_c), \end{aligned}\right\} \tag{4.1}$$

where P_A, P_B are power series in $y-y_c$, the leading terms of which are unity, and a subscript c denotes evaluation at $y = y_c$. The difficulty in using these, when one boundary is at infinity, arises from having to find a solution that remains bounded as $y \to \infty$. Since the point at infinity is an irregular singularity of the inviscid equation (3.1), the power series P_A and P_B are convergent only for $|y-y_c| < \infty$, and this is the cause of the trouble. Hughes & Reid (1965) have shown in the ordinary hydrodynamic case that this difficulty may be overcome for most profiles by obtaining another representation for Φ valid in the neighbourhood of $y = \infty$ and then matching the two solutions at some point in their common domain of validity. Their method, outlined below, remains valid for the hydromagnetic case under consideration here.

If the required inviscid solution is normalized by the condition $\Phi(y_c) = 1$, then it has to be of the form

$$\Phi(y) = K\phi_A(y) + \phi_B(y), \tag{4.2}$$

where ϕ_A and ϕ_B are given by (4.1) and K is a constant depending upon the parameters α, c and $\bar{Q}$. To find K so that Φ will remain bounded as $y \to \infty$, the independent variable in the inviscid equation is transformed to $w = e^{-y}$ thus mapping the point $y = \infty$ into the point $w = 0$. The resulting equation has the point $w = 0$ for a regular singular point with exponents $\pm\delta$ and is

$$(U-c)(w^2\ddot{\phi} + w\dot{\phi} - \alpha^2\phi) - U''\phi + i\alpha\bar{Q}\phi = 0, \tag{4.3}$$

a dot denoting differentiation with respect to w, and we have assumed that $U \to 1$ as $y \to \infty$. The solution of (4.3) regular in the neighbourhood of $w = 0$ is of the form

$$\Phi(y) = \bar{K}w^{\delta}P_{\infty}(w), \tag{4.4}$$

where $\bar{K}$ is a constant and $P_{\infty}(w)$ is a power series in w. This form of Φ is valid in the region $0 < y-y_c \leqslant \infty$. Thus there are now available two overlapping representations for Φ in the forms (4.2) and (4.4) and the constants K and $\bar{K}$ are determined by making Φ, Φ' continuous at some convenient point in their common domain of validity. If y_0 is such a point then the equations determining K, $\bar{K}$ are

$$\begin{aligned} K\Phi_A(y_0) + \Phi_B(y_0) &= \bar{K}e^{-\delta y_0}P_{\infty}(e^{-y_0}), \\ K\phi'_A(y_0) + \phi'_B(y_0) &= -\bar{K}e^{-\delta y_0}\{\delta P_{\infty}(e^{-y_0}) + e^{-y_0}\dot{P}_{\infty}(e^{-y_0})\}. \end{aligned} \tag{4.5}$$

For our profile the constant K can be calculated exactly. It is the coefficient of $y-y_c$ in the expansion of the regular part of $\exp\{-\delta(y-y_c)\}f(\xi)$, $f(\xi)$ being given by the equation (3.11). After a little calculation it is seen to be

$$K(\delta) = 1 - 2\gamma - \delta - \psi(p+1) - \psi(q+1), \tag{4.6}$$

where $\gamma = 0{\cdot}5772$ is Euler's constant.

In our calculations we used the form (4.6) for K. ϕ_A and ϕ_B were calculated using the method of Frobenius and we record below the first few terms in the series

$$\left.\begin{aligned} \phi_A &= (y-y_c) - \tfrac{1}{2}(1+iS)(y-y_c)^2 + \tfrac{1}{6}(1+\alpha^2+\tfrac{1}{2}iS-\tfrac{1}{2}S^2)(y-y_c)^3 + \ldots, \\ \phi_B &= 1 + \frac{2\alpha^2-2-7iS+3S^2}{4}(y-y_c)^2 + \ldots - (1+iS)\,\phi_A\log(y-y_c). \end{aligned}\right\} \tag{4.7}$$

In the expression for ϕ_B above $\log(y-y_c)$ is to be taken as $\log|y-y_c|-\pi i$ for $y-y_c < 0$ (see Stuart 1954).

After experimenting it was decided to retain terms up to $O(y_c^3)$ in ϕ_A and ϕ_B. We found that for $\bar{Q} = 0$ this gave a neutral stability curve virtually indistinguishable from the one calculated by Hughes & Reid (1965). For $\bar{Q} > 0$ the values of y_c are even smaller (see figure 5) on the neutral stability curves, so the approximation improves in accuracy.

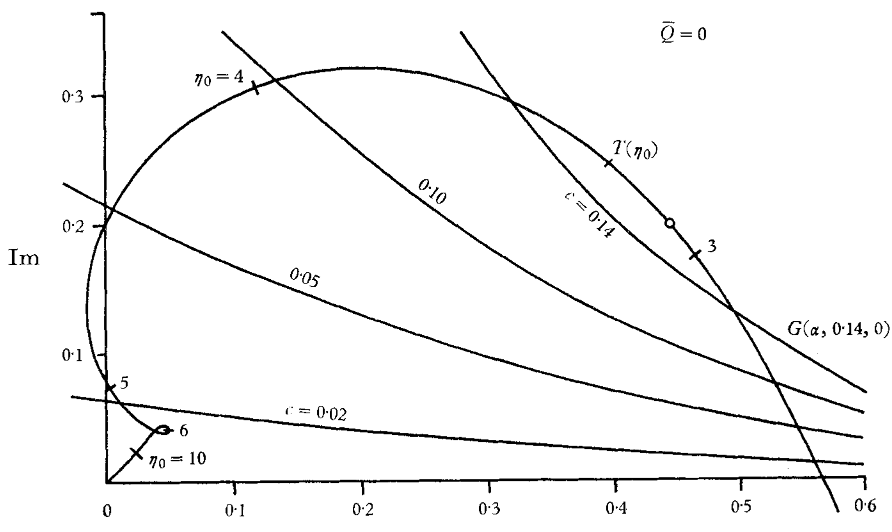

FIGURE 1

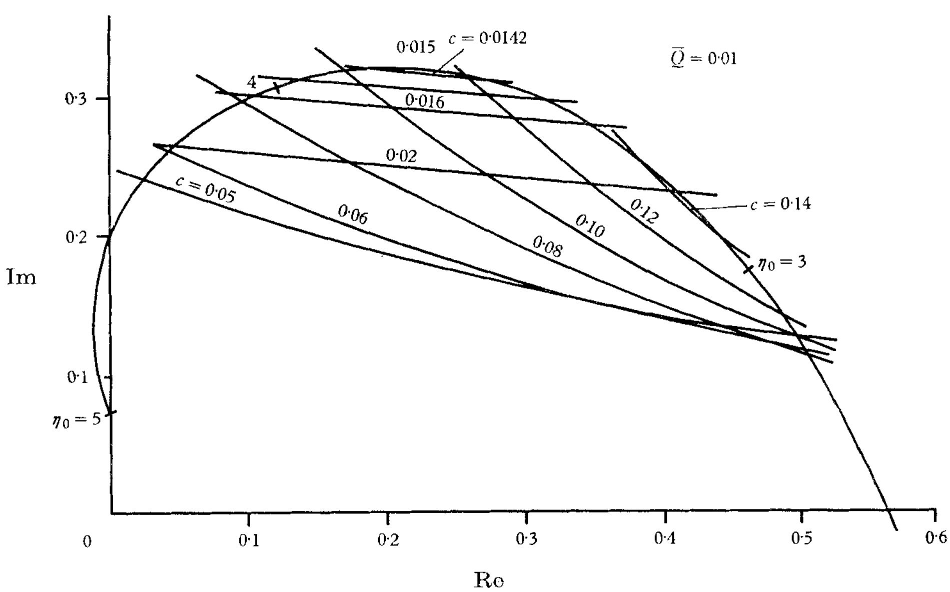

FIGURE 2

5. The characteristic equation

The required general solution of (3.1) can now be written in the form

$$\phi = C_1\Phi + C_2\chi_3(\eta), \tag{5.1}$$

where C_1, C_2 are constants and $\chi_3(\eta)$ is the viscous solution that is exponentially small for $|y-y_c| \gg |\alpha R U_c'|^{-\frac{1}{3}}$ (see Stuart 1954). Imposing the boundary conditions (2.9) at $y = 0$ and eliminating C_1, C_2 gives the characteristic equation in the form

$$\frac{\chi_3(\eta_0)}{\eta_0\chi_3'(\eta_0)} = \frac{\Phi(0)}{-y_c\Phi'(0)}, \tag{5.2}$$

where η_0 is the value of η for $y = 0$. In (5.2) a prime on the left-hand side denotes differentiation with respect to η and on the right it denotes differentiation with

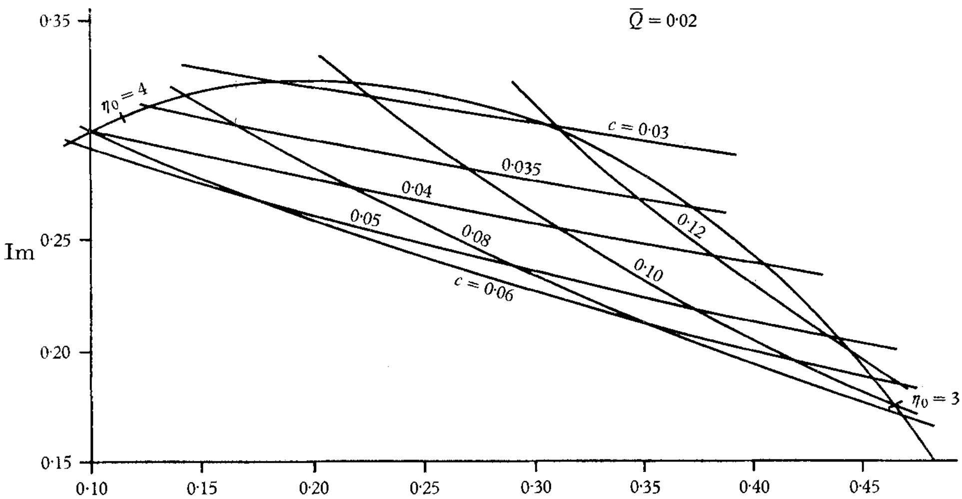

FIGURE 3

respect to y. The right-hand side of (5.2) depends only upon the inviscid solution and is a function of α, c and $\bar{Q}$, whereas the left-hand side depends only upon η_0. To emphasize this, we can write (5.2) as

$$T(\eta_0) = G(\alpha, c, \bar{Q}). \tag{5.3}$$

$T(\eta_0)$ is a universal function known as the Tietjens function and has been tabulated by several authors. In our numerical calculations we have used the tabulation due to Miles (1960).

Equation (5.3) was solved by the graphical method given by Tollmien (1929). The real part of T was plotted against its imaginary part giving a curve with parameter η_0. On the same diagram was plotted the real part of G against its imaginary part for a single value of $\bar{Q}$ and a series of values of c and α. Equation (5.3), being complex, represents in general two relationships and the inter-

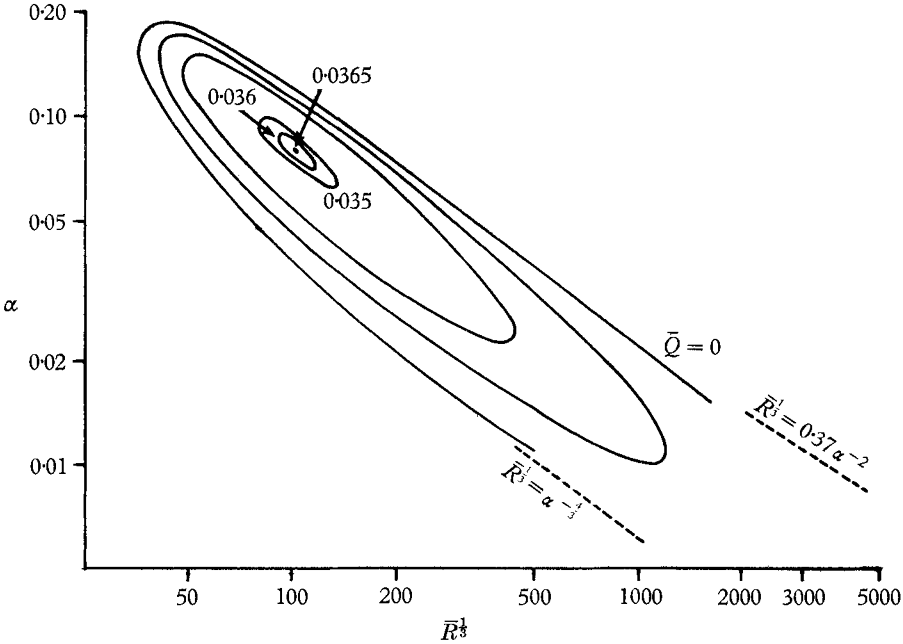

FIGURE 4. Curves of neutral stability.

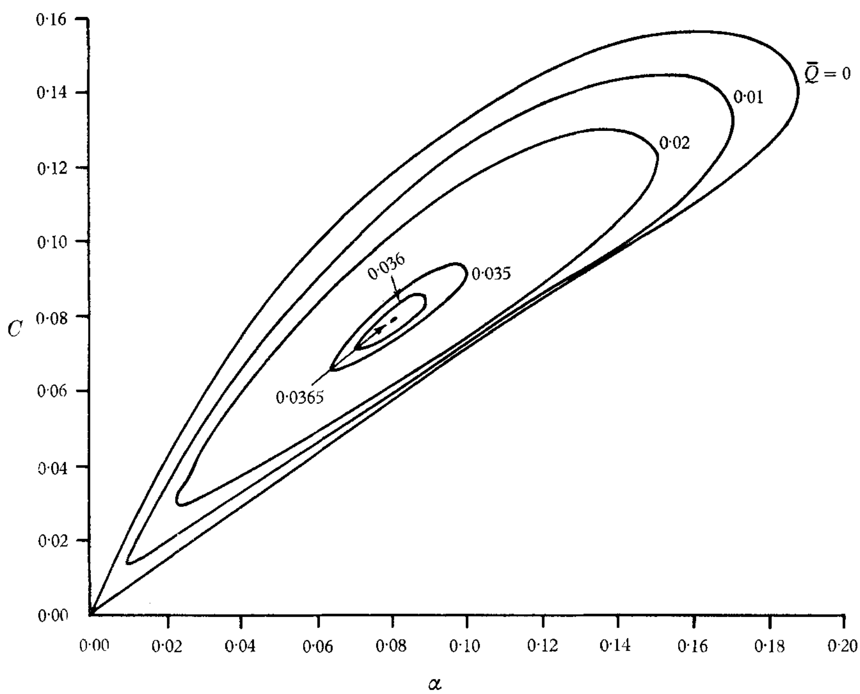

FIGURE 5. Curves of neutral stability.

section of the T and G curves therefore, in general, determines two sets of eigenvalues on the curve of neutral stability. The curves of neutral stability calculated for various values of $\bar{Q}$ are shown in figure 4. Figures 1–3, show the T and G curves for $\bar{Q} = 0$, 0·01, 0·02 respectively. For the other values of $\bar{Q}$ for which the neutral stability curves have been calculated the G curves follow the same pattern as for $\bar{Q} = 0{\cdot}01$ and $\bar{Q} = 0{\cdot}02$ but get progressively close to each other and are difficult to distinguish separately.

6. Results for two-dimensional disturbances

When there is no magnetic field present, i.e. $\bar{Q} = 0$, the G curves in figure 1 move steadily upwards and away from the T curve as c increases. The G curve tangential to the T curve determines a minimum value of the Reynolds number for which a real eigenvalue c exists. This is the critical Reynolds number R_{crit}. The neutral stability curve for $\bar{Q} = 0$ in figure 4, calculated from the intersections of T and G curves in figure 1 is in agreement with the one given by Hughes & Reid (1965). The points inside the curve correspond to unstable disturbances and those outside to the stable disturbances. The two branches of the neutral stability curve both tend to the $\bar{R}$ axis, and also α and c both tend to zero, as $\bar{R} \to \infty$. The asymptotic forms of the lower and upper branches of the curve respectively are (see Hughes & Reid 1965)

$$\bar{R}^{\frac{1}{3}} = \alpha^{-\frac{4}{3}}, \quad \bar{R}^{\frac{1}{3}} = 0{\cdot}37\alpha^{-2}.$$

When $\bar{Q} > 0$, the G curves (figures 2, 3) lie above and away from the T curve for sufficiently small values of c showing that the eigenvalue $c = 0$ does not exist. As c increases from zero, the G curves move towards the T curve but begin to move away again as c continues to increase. For $\bar{Q}$ greater than about 0·0365, no intersection between the G and T curves takes place. This behaviour of the G curves causes the neutral stability curves to be closed curves thus giving a minimum, as well as a maximum critical Reynolds number beyond which all two-dimensional disturbances are stable. As $\bar{Q}$ increases, the neutral stability curves shrink in area turning into a point for $\bar{Q} = 0{\cdot}0365$ and vanishing completely for values of $\bar{Q}$ greater than this.

The numerical analysis described above shows the behaviour of the inviscid part of the characteristic equation (5.2) for small values of α, c to be very different when $\bar{Q} > 0$ from when $\bar{Q} = 0$. This is caused by the change in the behaviour of its imaginary part and may be seen from the following.

For small values of α it can be shown by expanding (4.6) that $K(\delta) \sim 1/\delta$, and, if we also ignore terms $O(c \log c)$, then it can be shown that (5.3) reduces to

$$T(\eta_0) = 1 - \delta/c. \tag{6.1}$$

If $\bar{Q} = 0$, then $\delta = \alpha$, and the imaginary part of the right-hand side of (6.1) is zero. But when $\bar{Q} \neq 0$, then, when $c \to 0$ for a fixed value of α, the imaginary part tends to infinity. Now the imaginary part of the Tietjens function $T(\eta_0)$ has a maximum value of about 0·32, so that for sufficiently small values of c (6·1) has no solution. The equation that determines the eigenvalue $\bar{R}$ on the curves of neutral stability is

$$\bar{R}^{\frac{1}{3}} = \frac{\eta_0}{y_c\{\alpha(1-c)\}^{\frac{1}{3}}},$$

and since c cannot be zero y_c is also not zero, and so $\bar{R}$ cannot become infinite. We deduce therefore that the curves of neutral stability are closed for all $\bar{Q} > 0$.

7. Correction for three-dimensional disturbances

It is only when two-dimensional disturbances are considered that we obtain this rather curious result that instability occurs for a finite range of the Reynolds number. We now show how the results obtained should be modified to give the minimum critical Reynolds number of all disturbances. The method was pointed out by Hunt (1966); and for a detailed explanation the reader must refer to his paper.

By choosing to consider an equivalent two-dimensional problem to the one posed by (2.7) we effectively reduce the parameters R and Q for a given flow by a factor $\cos\hat{\theta}$, $\hat{\theta} = \tan^{-1}(\alpha_3/\alpha_1)$ being the angle made by the disturbance with the flow direction. So that the relationships between R, Q the given parameters of a real flow and $\bar{R}$, $\bar{Q}$ the parameters artificially introduced for the purpose of simplifying the analysis are

$$\bar{R} = R\cos\hat{\theta}, \quad \bar{Q} = Q\cos\hat{\theta}. \tag{7.1}$$

From the neutral stability curves in figure 4 we deduce a relationship between $\bar{Q}$ and $\bar{R}$ of the type

$$\bar{R}_{\text{crit}} = \mathscr{F}(\bar{Q}), \tag{7.2}$$

giving the critical Reynolds number for the two-dimensional disturbances. This is shown as the curve $\hat{\theta} = 0$ in figure 6.

Substitution from (7·1) into (7.2) gives the equation

$$R_{\text{crit}} = \frac{\mathscr{F}(\bar{Q}\cos\hat{\theta})}{\cos\hat{\theta}}, \tag{7.3}$$

from which the relationship between Q and R_{crit} for any value of $\hat{\theta}$ may be calculated. We have done this for $\hat{\theta} = 30°$, $60°$ and the result is shown in figure 6. Observe that these curves start above the curve $\hat{\theta} = 0$ but eventually cross over and lie below, giving, after a certain value of $\bar{Q}$, a value for the critical Reynolds number which is less than the corresponding value on the curve $\hat{\theta} = 0$. The true critical Reynolds number is given by the curve $\hat{\theta} = 0$, only for values of $Q < Q_0$ where Q_0 is the abscissa of the point where the tangent from the origin touches the curve $\hat{\theta} = 0$. For values of $Q > Q_0$ the points on the tangent determine the minimum value. The R_{crit} curve, taking into account all disturbances, is shown by the thick lines in figure 6. Hunt (1966) has shown that for $Q > Q_0$ the most unstable disturbances travel at an angle θ^* with the flow direction such that

$$\cos\theta^* = Q_0/Q.$$

In our particular case we found Q_0 to be about 0·017, so that this relationship becomes

$$\cos\theta^* \doteqdot \frac{0{\cdot}017}{Q}. \tag{7.4}$$

In figure 7, we have plotted R_{crit} against the parameter M where

$$M = \sqrt{(RQ)} = \sqrt{(A^2RR_m)} = r\overline{H}_1,$$

and r depends upon the fluid properties only.

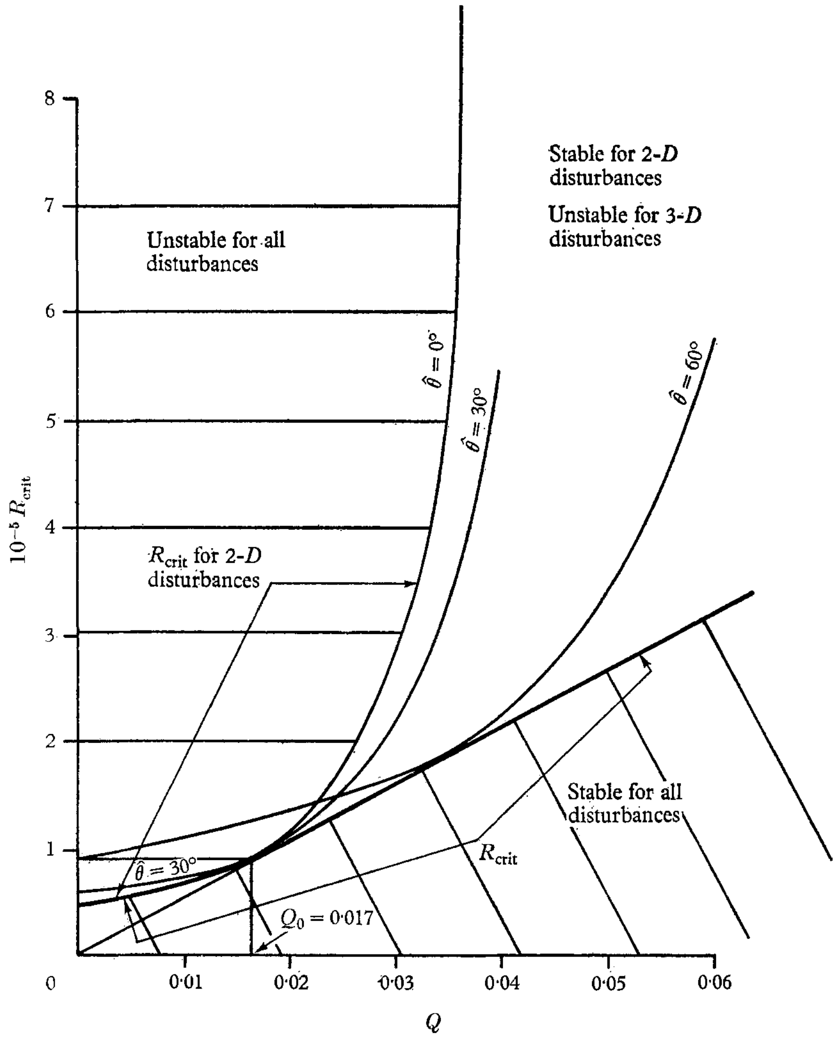

FIGURE 6. Stability limits in terms of Reynolds number and Q.

Since M is proportional to the applied parallel magnetic field, the (R_{crit}, M)-curves are much more instructive of the effect of the magnetic field than the (R_{crit}, Q)-curves (Q contains a velocity term). For large values of M the relationship between R_{crit} and M is

$$R_{crit} \doteqdot 2{\cdot}3 \times 10^3 M. \tag{7.5}$$

For a given value of M the angle θ^* at which the most unstable disturbance travels is given by

$$\cos\theta^* \doteqdot \frac{40}{M}. \tag{7.6}$$

We may note as an example that when $M = 80$ the most unstable disturbance makes an angle of 60° with the flow direction.

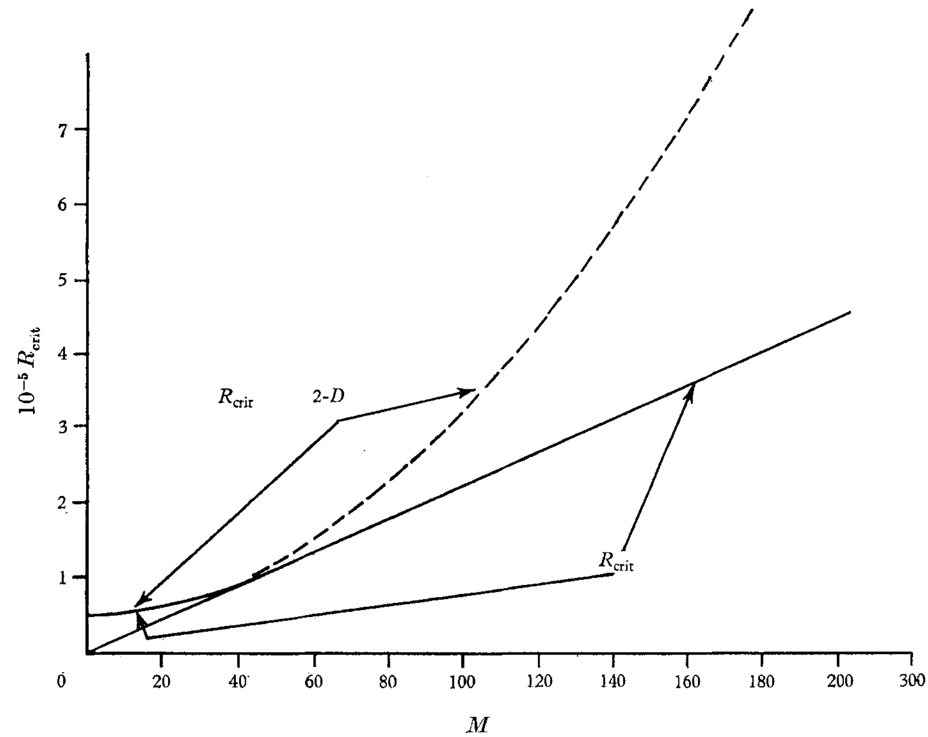

FIGURE 7. Stability limits in terms of Reynolds number and M.

8. Concluding remarks

The exponential profile occurring in this problem has been analysed for stability in the non-conducting case by several authors. It was Freeman (see Chiarulli & Freeman 1948) who showed that with this profile the inviscid form of the Orr–Sommerfeld equation can be solved exactly by transforming it to a hypergeometric equation. The transform we effected in our case was suggested by his. The value of the exact solution lies in the understanding it provides of the differences in the analytical structure of the inviscid equation in the conducting and non-conducting cases. Thus we are able to explain the closure of our neutral stability curves, and, though we have not made it explicit in this paper, we feel that considerably more insight is to be gained through the study of this exact solution.

It would be interesting to explain the physical mechanism which causes the three-dimensional disturbances to play a different role in the conducting flows, and particularly for the existence of Q_0, the value of Q after which these disturbances become important. But the author has not been able to do so satisfactorily and leaves this to someone with greater physical insight.

The investigation reported here formed part of a Ph.D. thesis submitted in the University of London, in March 1967. The author is greatly indebted to Dr D. H. Michael of University College London, who suggested the problem and who throughout this research gave helpful guidance. Some of this work grew from a fruitful conversation with Mr J. C. R. Hunt of the Central Electricity Generating Board, and the author wishes to record his gratitude for this. Thanks are due to Miss S. Burrough of University College London, who helped with a great deal of the numerical work involved. Finally, the author wishes to record his deeply felt gratitude to the Science Research Council who provided sustenance during the author's period of Research Studentship in the form of a grant.

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